9^{th} World Mathematics Team Championship 2018

Intermediate Team Detailed Solutions

1. The set {1, 2, 3, 5, 8} has the desired property. Assume there exists a set $A = \{a_1, a_2, a_3, a_4, a_5\}$, with $a_5 < 8$ the largest, satisfying the property. If $1,2 \in A$ then among the remaining three numbers there are no consecutive numbers. The only such case is $3, 5, 7 \in A$ and then 1+7=3+5. So at least one of 1 and 2 is not an element of A. By the similar arguments at least one of 7 and 6 is not an element of A. Then $3,4,5 \in A$ and easy verification of all 4 cases leads to a contradiction.

Answer: 8

2. Note that $2018 = 2 \times 1009$ and 1009 is a prime number. Also for $n = 1\underbrace{999...99}_{112}$ we have $S(n) = 1009$ and $S(n + 1) = 2$. The sum of the digits of $n - 1 = 1\underbrace{999...98}_{112}$ is 1008, so $n – 1$ is not a solution. Assume $S(m)S(m + 1) = 2018$ for $m < n – 1$. The sum of the digits of any positive integer, less than $n – 1$ is less than 1009. So both $S(m)$ and $S(m + 1)$ are less than 1009, a contradiction.

Answer: $n = 1\underbrace{999...99}_{112}$.

3. Denote $S_{ABD} = x$, $S_{BDX} = S_{ADY} = y$ and $S_{DXY} = z$. We have

$$x + y = S_{ABX} = \frac{BX}{BC} \times S_{ABC} = \frac{4}{6} \times 18 = 12 \text{ and } y + z = S_{AXY} = S_{ABC} - 12 - 2 = 4.$$

We use that $x = 12 – y$, $z = 4 – y$ and $y^2 = x \times z$. Thus $y^2 = (12 – y)(4 – y)$ which implies $y = 3$. Thus $x = 9$ cm^2.

Answer: 9

4. Denote the number of words of length *n* with the given property by T_n.

Partition the words from T_n in two sets: words with last letter *b* (denote this set by T_n^b) and words with last letter *a* (denote this set by T_n^a). It is clear that $T_n^b = T_{n-1}$ and $T_n^a = T_{n-2}$. So $T_n = T_{n-1} + T_{n-2}$. From $T_1 = 2$ and $T_2 = 3$ we obtain: $T_3 = 5$, $T_4 = 8$, $T_5 = 13$, $T_6 = 21$ and $T_7 = 34$.

Answer: 34

5. Assume all cells have odd number of adjacent cells. By considering cells (1,1) and (2,2) it is easy to see that there are even number of colored cells among (3,2) and (2,3). The same arguments for (4,4) and (5,5) imply that there are even number of colored cells among (3,4) and (4,3). Now the cell (3,3) has even number of colored adjacent cells. The example of exactly one cell with even number of colored adjacent is given by the table.

X	X		X	X
		X		
	X	X	X	
		X		
X	X		X	X

Answer: 1

6. The first minute each elephant drank 4 liters; the second minute each elephant drunk 5 liters and the third minute each one drunk 8 liters. Each of the last 5 has 4 + 5 + 8 =17 liters.

Answer: 17

7. For 1 min the big clock arrow travels for halve degree. For 1 min the small clock arrow travels for 6 degrees. So the angle between big clock arrow and 12 o'clock direction equals $210° + 19° = 229°$. The angle between small clock arrow and 12 o'clock direction equals $38 \times 6 = 228°$. Then the angle between two arrows equals $1°$.

Answer: 1

8. It follows from isosceles triangle *AMN* and $\angle BAC = 20°$ that $\angle AMN = 20°$. Then $\angle CNM = 40°$ and *NM* = *MC* implies $\angle NCM = 40°$. From triangle *ANM* we obtain $\angle CMN = 100°$. Now $\angle BMC = 60°$ and triangle MBC is equilateral. Therefore *MB* = *MC* = *MN* and it follows from isosceles triangle *MBN* that $\angle BNM = 10°$.

Answer: 10

9. For $x \in \left[\frac{1}{2}; \frac{2}{3}\right]$ the equation is equivalent to $(2x-1)-(3x-2)=a-x$ or $a=1$.

Answer: 1

10. ***T* = 1.** Since $\overline{c1b}$ is divisible by 5 we have that $b = 0$ or $b = 5$. But $b \neq 0$ thus $b = 5$. Since $\overline{15c}$ is divisible by 4 we have that $\overline{5c}$ is divisible by 4, so $c = 2$ or $c = 6$.

Since $\overline{521}$ is not divisible by 3 we have $c = 6$.

Answer: 6

11. ***T* = 1.** Write the equation $x + y = xy - 7$ in the form $(x - 1)(y - 1) = 8$.

Since x is positive integer and $x - 1$ is a divisor of 8 it can take 4 values: 2, 3, 5 and 9. The corresponding values of y are: 9, 5, 3 and 2.

Answer: 4

12. ***T* = 12.** The number of games equals 21. Each game contributes to the total number of points by 3 or 2. Therefore the total number of points can be any number between 42 and 63. Since it is divisible by 24 it equals 48. Then the number of wins is 48 – 42 = 6.

Answer: 6

13. ***T* = 9.** Note that 111 and 111111 are both divisible by 3 but not by 6. It is easy to see that 111111111 is divisible by 9 but not by 27. Using the above, $T = 9$ and

$$3 \times 33 \times 333 \times \ldots \times \underbrace{33\ldots33}_{9} = 3^9 \times 1 \times 11 \times 111 \times \ldots \times \underbrace{11\ldots11}_{9}$$

we conclude that $k = 13$.

Answer: 13

14. ***S* = 17, *T* = 10.** Denote the speed of the student by x and the speed of the tram by y. If the distance between two trams is D then $\frac{D}{x+y} = T = 10$ and $\frac{D}{y-x} = S = 17$. Therefore $\frac{x+y}{y-x} = \frac{17}{10}$ which is equivalent to $27x = 7y$ or $\frac{x}{y} = \frac{7}{27}$.

Answer: $\frac{7}{27}$