9^{th} World Mathematics Team Championship 2018

Intermediate Level Round 1 Numerical Answers

1	2	3	4	5	6	7	8
B) 72	A) 1	C) 301	D) –1	C) 40	D) 13	E) 12	E) 85

9	10	11	12	13	14	15
B) 6	C) 4	A) 0	B) 2	D) 54	D) 81	C) 36

Intermediate Level Round 1 Detailed Solutions

1. Since $38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$ the number of positive integer divisors equals

$$(3+1)\times(2+1)\times(2+1)\times(1+1)=72$$

2. Since $2^{x+1} \cdot 4^{x+2} \cdot 8^{x+3} = 2^{x+1+2(x+2)+3(x+3)} = 2^{6x+14}$ and $16^{x+4} = 2^{4(x+4)} = 2^{4x+16}$ we have solve $6x + 14 = 4x + 16$. The solution is $x = 1$.

3. Since

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{100^2}\right)$$

$$=\frac{(1\times3)\times(2\times4)\times(3\times5)\times(4\times6)\times\dots\times(98\times100)\times(99\times101)}{(100!)^2}=\frac{101}{200}$$

we have a = 101 and b = 200 giving $a + b$ = 301

4. We have $\frac{1+\sqrt{12}}{1-\sqrt{12}}+\frac{\sqrt{48}}{\sqrt{121}}+\frac{2}{11}=\frac{1+2\sqrt{3}}{1-2\sqrt{3}}+\frac{4\sqrt{3}+2}{11}=-\frac{13+4\sqrt{3}}{11}+\frac{4\sqrt{3}+2}{11}=-1.$

5. We use the well-known fact that $S_{ABO}\times S_{CDO}=S_{BCO}\times S_{ADO}$.

Therefore $S_{ADO}=\frac{S_{ABO}\times S_{CDO}}{S_{BCO}}=\frac{35\times 48}{42}=40$ cm^2.

6. Since $(a+bx)^3+(a-bx)^3=2a^3+6ab^2x^2$ it follows that $2a^3=-16$ and $6ab^2=-108$.

The first equation implies $a = -2$ and the second one $b = 3$, so $a^2 + b^2 = 13$.

7. The shortest altitude is perpendicular to the side of 21 cm. By error and trial, having in mind $5^2 + 12^2 = 13^2$ and $12^2 + 16^2 = 20^2$, we see that the answer is 12 cm.

8. The smallest 3-digit number divisible by 7 equals $105=7\times 15$ and the largest 3-digit number divisible by 7 equals $994=7\times 142$. Thus the number of 3-digit numbers divisible by 7 equals 142 – 14 = 128.

The smallest 3-digit number divisible by 21 equals $105=21\times 5$ and the largest 3-digit number divisible by 21 equals $987=21\times 47$. Thus the number of 3-digit numbers divisible by 21 equals 47 – 4 = 43.

The number of 3-digit numbers divisible by 7 but not divisible by 3 equals 128 – 43 = 85.

9. It is not possible to put *a* or *b* after *ab*. So, after first *a* in *w* we have at most 2 letters. Since the longest word consisting only by letter *b* has length 3 we conclude that the length of *w* is at most 6. The example of such word is *bbbaaa*.

10. Since $2x^4 - 2x^2y^2 + y^4 - 8x^2 + 20 = (x^2 - y^2)^2 + (x^2 - 4)^2 + 4$ we conclude that the least value of the given expression is at least 4. It is achievable for $x = y = 2$.

11. $2018^3 - 2019^3 + 1 + 3\times 2018\times 2019$

$= 1 + 3\times 2018\times 2019 - (2019^3 - 2018^3)$

$= 1 + 3\times 2018\times 2019 - \left(2019^2 + 2019\times 2018 + 2018^2\right)$

$= 1 + 2\times 2018\times 2019 - 2019^2 - 2018^2$

$= 1 - (2019 - 2018)^2 = 0$.

12. Since $S_{APD} = S_{ABQ} = \frac{1}{2} S_{ABCD}$ we have 7+3 = 8 + x, thus x = 2 cm^2.

13. Note that $3 - 3^2 + 3^3 - 3^4 = 60$ which implies that $3^n(3 - 3^2 + 3^3 - 3^4)$ is divisible by 60 for any positive integer n. Modulo 60 the sum reduces to $3^{2017} - 3^{2018}$. Since 3^{4k+1} equals 1 modulo 60 we have that $3^{2017} - 3^{2018}$ modulo 60 equals –6 or the remainder is 54.

14. Since $A = (3-2)\times A = \frac{3^{64} - 2^{64}}{3^{60}} = 81 - \frac{2^{64}}{3^{60}}$ and

$$\frac{2^{64}}{3^{60}} < \frac{1}{2} \Leftrightarrow 2^{65} < 3^{60} \Leftrightarrow 2^{13} < 3^{12} \Leftrightarrow 8\times 2^{10} < 9\times 3^{10}$$

we infer that the closest integer to A is 81.

15. It follows from $1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ that

$$10^3 + 11^3 + ... + 20^3 = \left(\frac{20\times 21}{2}\right)^2 - \left(\frac{9\times 10}{2}\right)^2 = 210^2 - 45^2 = 165\times 255 = 3^3\times 5^2\times 11\times 17$$

and the sum of all prime divisors equals 3 + 5 + 11 + 17 = 36.