9^{th} World Mathematics Team Championship 2018

Advanced Team Detailed Solutions

1. Since the leading coefficient is 1 the root is an integer number. Further, it follows from $a=-x^2+2x-4-\frac{7}{x+2}$ that $x + 2$ divides 7, i.e. $x = -1, -3, 5$ and -9. The corresponding values of a are –14, –12, –20 and –102. The least value of a is –102 and then $a^2 = 10404$.

Answer: 10404

2. If x is a root then $-x$ is also a root. To have 5 roots we need a root $x = 0$. Then $b=-a^2$ or $b=a^2+2$. In the first case the equation has three roots 0 and $\pm\sqrt{2}$. In the second case there are five distinct roots: 0, $\pm\sqrt{2}$ and $\pm\sqrt{2a^2+4}$.

Now the minimum value of $2a+|b|+49=a^2+2a+51$ is 50 and is attained for $a = -1$.

Answer: 50

3. Since $\left(1\pm\sqrt{2}\right)^3=7\pm5\sqrt{2}$ we have $a=\sqrt[3]{7+\sqrt{2}}+1$ and $b=\sqrt[3]{7+\sqrt{2}}-1$.

Now $a - b = 2$ and after substitution we obtain the answer 7.

Answer: 7

4. The centers of the four balls are vertices of regular pyramid with edge length equal to the diameter $d=40\left(3-\sqrt{6}\right)$ of the ball. The altitude of such a pyramid is $\frac{d\sqrt{6}}{3}$.

The distance from the ground to the top equals

$$\frac{d\sqrt{6}}{3}+d=40\left(3-\sqrt{6}\right)\left(\frac{\sqrt{6}+3}{3}\right)=40 \text{ cm.}$$

Answer: 40 cm

5. Let $1\le a_1<a_2<...<a_k$ be the elements of *A*. By setting $x = a_1$ and $y = a_2$ we obtain $\frac{1}{a_2}\le\frac{1}{a_1}-\frac{1}{25}\le\frac{24}{25}$, i.e. $a_2\ge 2$. Analogously $a_3\ge 3$, $a_4\ge 4$, $a_5\ge 5$, $a_6\ge 7$, $a_7\ge 10$, $a_8\ge 17$, $a_9\ge 54$ and $a_{10}<0$, a contradiction. The set {1, 2, 3, 4, 5, 7, 10, 17, 54} has the required property.

Answer: 9

6. Each of the first 5! = 120 numbers (having altogether 1200 digits) begins with 10234. The last number is 1023498765. Thus, the 1198-th digit equals 7.

Answer: 7

7. For $x = 0$ the inequalities become $b \ge 1 \ge a$. Further, it follows from $ax^2 + bx + 1 \ge bx^2 + x + a$ that $a\ge b$. Finally, $a = b = 1$ and there is only one pair.

Answer: 1

8. Easy induction shows that for the set $\{1, 2, 3,\dots, n, n + 6\}$ we have $S=(n+1)!-1$. Therefore $S + 1 = 100!$ and the largest prime divisor of 100! is 97.

Answer: 97

9. All considered triples have equal products. It is easy to see that if $1=d_1<d_2<...<d_{27}=900$ are all divisors of 900 then the desired number equals

$\tau(d_1)+\tau(d_2)+...+\tau(d_{27})$ where $\tau(n)$ is the number of divisors of *n*. Since $900 = 2^2\cdot3^2\cdot5^2$ we have $\tau(d_1)+\tau(d_2)+...+\tau(d_{27})=(1+2+3)^3=216$.

Answer: 216

***T* = 97**

10. If D is the symmetric point of C with respect to M then $\angle NMK = \angle QDP$.

If $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle ACB = \gamma = T°$ then $\angle ADP = \frac{180° - \beta}{2}$ and $\angle BDQ = \frac{180° - \alpha}{2}$. It follows that $\angle QDP = \angle ADP + \angle BDQ - \angle ADB = \frac{180° - \gamma}{2}$, i.e. $2\angle NMK = 83°$.

Answer: 83

***T* = 7**

11. $3\binom{9}{4} = 378$. In general the answer is $3\binom{T+2}{4}$

Answer: 378

***T* = 1**

12. The equality is equivalent to (x – 7)(y + 7) = –41 and the greatest value is attained for x = –34 and y = –6.

Answer: 204

***T* = 50**

13. A person climbs a moving escalator. From the moment he steps on the escalator to the moment he reaches the end of the escalator he climbs T steps. When climbing two times faster from the moment he steps on the escalator to the moment he reaches the end of the escalator he climbs T + 10 steps. Find the number of steps the escalator in rest has.

Solution: $\frac{50.60}{2.50 - 60} = 75$ In general the answer is $\frac{pq}{2p - q}$ where p and q are the given number of steps.

Answer: 75

***T* = 9, *S* = 7**

14. Denote the tangent point of k_1 and k_2 by I. Since $\angle IAC = \frac{IC}{2} = \frac{IA}{2} = \angle IAB$ we have that IA is angular bisector of angle BAC. The same is true for IC, ID and IB and corresponding angles. Therefore $ACDB$ is circumscribed, giving $AC + BD = 2AB$. But $AB = 2\sqrt{r_1 r_2} = 6\sqrt{7}$ (it follows from O_1O_2BA) and thus $(AC + BD)^2 = 4AB^2 = 1008$.

Answer: 1008