9th World Mathematics Team Championship 2018

Advanced Level Team Round

English Version

Instruction: This round has 14 questions (**40 minutes**).
Each question is worth 5 points.
No point penalty for submitting wrong answer.

1. Let a be the least integer for which the equation $x^3 + ax + 2a + 15 = 0$ has rational root. Find a^2.

2. Let a and b be real numbers such that the equation $\left|\left|x^2 - 1\right| - b\right| = a^2 + 1$ has exactly 5 distinct real roots. Find the least value of $2a + |b| + 49$.

3. If $a = \sqrt[3]{7} + \sqrt[3]{7 + 5\sqrt{2}}$ and $b = \sqrt[3]{7} - \sqrt[3]{7 - 5\sqrt{2}}$ find the value of $a^2 + b^2 - 2ab + 3$.

4. The diameter of a football (soccer) ball is $40\left(3 - \sqrt{6}\right)$ cm. Three football balls lay on the ground and each one touches the other two. Another football ball is put on the top of given balls such that it touches each of the three balls. Find the distance from the ground to the highest point of the given "pyramid" of four balls.

5. Let A be a set of positive integers such that for any two distinct x and y from A we have:
$\left|\frac{1}{x} - \frac{1}{y}\right| \geq \frac{1}{25}$. Find the maximum possible number of elements of A.

6. All 10 digit numbers with distinct digits are written one after another in increasing order. Find the 1198-th digit in the sequence obtained.

7. Find the number of ordered pairs (a,b) of real numbers such that such that:

$$x^2 + ax + b \geq ax^2 + bx + 1 \geq bx^2 + x + a \qquad \text{for all real numbers } x.$$

8. For every nonempty subset A of the set $\{1, 2, 3, \ldots, 99, 105\}$ denote by $P(A)$ the product of elements of A. If S is the sum of all such products find the largest prime divisor of $S + 1$.

9. All ordered triples of positive integers (a, b, c) are written one after another by the following rules:

1. If $a_1b_1c_1 < a_2b_2c_2$ then (a_1, b_1, c_1) is before (a_2, b_2, c_2).
2. If $a_1b_1c_1 = a_2b_2c_2$ and $a_1 < a_2$ then (a_1, b_1, c_1) is before (a_2, b_2, c_2).
3. If $a_1b_1c_1 = a_2b_2c_2$ and $a_1 = a_2$ and $b_1 < b_2$ then (a_1, b_1, c_1) is before (a_2, b_2, c_2).

How many triples are there between (1,1,900) and (900,1,1) inclusive?

***T* = the answer of problem #8**

10. In a triangle ABC with $\angle ACB = T^\circ$ points P and Q on the side AB are such that $AP = BC$ and $BQ = AC$. If M, N and K are the midpoints of AB, CP and CQ respectively find $2\angle NMK$ in degrees.

***T* = the answer of problem #6**

11. Equilateral triangle of side length T is divided into equilateral triangles of side length 1 by lines parallel to its sides. Find the number of parallelograms bounded by the segments of the grid.

***T* = the answer of problem #7**

12. Find the greatest value of xy where x and y are integers such that

$$(T + 6)(x - y) + xy = 8.$$

***T* = the answer of problem #2**

13. A person climbs an escalator that is moving in the same direction. From the moment he steps on the escalator to the moment he reaches the end of the escalator he climbs T steps. When climbing two times faster from the moment he steps on the escalator to the moment he reaches the end of the escalator he climbs T+10 steps. Find the number of steps the escalator in rest has.

***S* = the answer of problem #3**

***T* = the answer of problem #5**

14. Two circles k_1 and k_2 of radii S and T, respectively, are externally tangent. If AB and CD are common tangents to k_1 and k_2 ($A, C \in k_1$ and $B, D \in k_2$) find $(AC + BD)^2$.