9^{th} World Mathematics Team Championship 2018

Advanced Level Round 2 Numerical Answers

1	2	3	4	5	6	7	8
35	1	3	2	39	3510	51	26

Advanced Level Round 2 Detailed Solutions

1. Let $n = 2^{\alpha} \times 3^{\beta} \times m$ where $\alpha \geq 0$ and $\beta \geq 0$. Then $d(2n) = (\alpha+2)(\beta+1)d(m) = 28$ and $d(3n) = (\alpha+1)(\beta+2)d(m) = 30$. Since gcd(28,30) = 2 we have $d(m) = 1$ or $d(m) = 2$.

If $d(m) = 2$ we obtain $(\alpha+2)(\beta+1) = 14$ and $(\alpha+1)(\beta+2) = 15$ with no solution.

If $d(m) = 1$ we obtain $(\alpha+2)(\beta+1) = 28$ and $(\alpha+1)(\beta+2) = 30$ with $\alpha = 5$ and $\beta = 3$. Therefore $d(6n) = (\alpha+2)(\beta+2) = 35$.

2. Since $\min\left\{x^2 - 2x + 5\right\} = 4$ (the minimum is achieved for $x = 1$) we have

$$\max\left\{\log_{\frac{1}{2}}\left(x^2 - 2x + 5\right)\right\} = -2 \text{ and } \min\left\{\left(\frac{1}{3}\right)^{\log_{\frac{1}{2}}\left(x^2 - 2x + 5\right)}\right\} = 9.$$

Also $\max\left\{9 - |x-1|\right\} = 9$ for $x = 1$. Therefore $x = 1$.

3. Since $\angle BPM = \angle BCM = 90°$ the quadrilateral $PBCM$ is cyclic. Then $\angle PCB = \angle PBM = \angle PCM = 45°$. For $\varphi = \angle CBM$ the sine theorem for triangle PBC implies:

$$\frac{PC}{BC} = \frac{\sin\angle PBC}{\sin\angle BPC} = \frac{\sin(\varphi + 45°)}{\sin(90° - \varphi)} = \frac{\sin(\varphi + 45°)}{\cos(\varphi)} = \frac{\sqrt{2}}{2}\tan\varphi + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{4}$$

For BC = 1 we have $PC = \frac{3\sqrt{2}}{4}$ and therefore $\frac{PC}{PA} = \frac{\frac{3\sqrt{2}}{4}}{\sqrt{2} - \frac{3\sqrt{2}}{4}} = 3$.

4. Denote $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + ...}}}}$. The $x^2 = 2 + x$ with roots $x = 2$ and $x = -1$. Since $x > 0$ we obtain $x = 2$.

5. Since $AC \perp BD$ we have that $AB^2 + CD^2 = BC^2 + AD^2$. Therefore

$AD^2 = AB^2 + CD^2 - BC^2 = 60^2 + 25^2 - 52^2 = 39^2$ and thus AD = 39 cm.

The diagonals AC and BD of quadrilateral $ABCD$ are perpendicular. If AB = 60 cm, BC = 52 cm and CD = 25 cm find AD.

6. Denote the number by $\overline{a_1a_2a_3a_4a_5}$. The number of solutions of

$a_1 + a_2 + a_3 + a_4 + a_5 = 22$ for $a_i > 0$ equals to $\binom{21}{4} = 5985$. From this number we have to subtract the number of solutions having $a_i > 9$ for some i = 1,2,3,4,5 (it is not possible to have $a_i > 9$ and $a_j > 9$ because then $a_1 + a_2 + a_3 + a_4 + a_5 > 22$). Replacing a_i by $a_i - 9$ leads to $a_1 + a_2 + a_3 + a_4 + a_5 = 13$ which admits $\binom{12}{4} = 495$

The answer is $5985 - 5 \times 495 = 3510$.

7. Denote by a, b, c, d and e the number of problems solved from day 1 to 5; 6 to 11, 12 to 14, 15 to 25 and 26 to 31, respectively. The condition of the problem implies

$a + b + c + d = n$, $a + b + c = 70$, $b + c + d = 80$ and $c + d + e = 60$. Therefore

$n + c = (a + b + c) + (c + d + e) = 130$, thus $n = 130 - c$. Since $b + c \leq 70$ it follows that $d \geq 10$ and $c \leq 50$. It is easy to see that any $c = 0, 1, \ldots, 50$ gives a solution. The answer is 51.

8. The balls can be placed on the circle by $\binom{10}{5}$ ways. Consider an arrangement that doesn't change after a rotation of $36k$ degrees where k, $(1 \leq k \leq 9)$ is a positive integer. Then 10 is a factor of $5k$ implying that k is even. It is easy to see that this is possible only for alternating colors (there are two such arrangements). Then all remaining $\binom{10}{5} - 2$ arrangements are grouped by 10's. The answer is $\frac{\binom{10}{5} - 2}{10} + 1 = 26$.